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Giant acceleration in slow-fast space-periodic Hamiltonian systems

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The motion of an ensemble of particles in a space-periodic potential well with a weak wavelike perturbation imposed is considered. We found that slow oscillations of the wave number of the perturbation lead to the occurrence of directed particle current. This current is amplified with time due to the giant acceleration of some particles. It is shown that giant acceleration is linked to the existence of resonant channels in phase space.

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In recent years considerable interest has been devoted to the ratchet effect: the generation of directed particle current in the absence of any biased forces. This phenomenon is relevant to a wide range of applications, including controlled photocurrents in semiconductors [1,2], motion of cold atoms in optical lattices [3–5], transport of passive tracers in meandering jet flows in the ocean [6], and biological and chemical systems (see [7] for a comprehensive review).

The ratchet effect in space-periodic Hamiltonian systems is associated with the asymmetry of the chaotic region in phase space [8–10]. Recently a new type of Hamiltonian ratchets was reported, in which a periodic potential is subjected to a sum of external forces which are periodic in time and coordinates [11]. Each of the forces induces strong but local chaotic diffusion in certain areas of phase space. This effect is achieved by means of resonance between temporal and spatial oscillations of the perturbation imposed. This resonance is asymmetric in momentum space, which provides asymmetry of crossing the separatrix and occurrence of directed transport. Combining such forces, we can make finite motion unstable for all ranges of the particle energy. In this way, even a weak perturbation of the potential can activate ballistic current of particles with the lowest initial energies. A similar effect was used in [12] in order to produce surfatron acceleration.

In the present paper we demonstrate the mechanism providing simultaneously generation and *giant acceleration* of directed current by means of a weak external perturbation. The possibility of giant acceleration arises due to adiabatic variations of the perturbation.

Consider an ensemble of noninteracting unit-mass point particles driven by a wavelike external force. The motion of each particle is described by the Hamiltonian

$$H = \frac{p^2}{2} - \cos x + \varepsilon \cos(\tilde{k}x + \nu t), \tag{1}$$

where $\varepsilon \ll 1$ and the wave number of the perturbation $\tilde{k}$ is a slowly varying parameter

$$\tilde{k} = k(1 + a \cos \Omega t), \quad |a| < 1, \quad \Omega \ll 1. \tag{2}$$

Physically this condition corresponds to slow libration of the external force with respect to axis x. Particle trajectories obey the Hamiltonian equations

$$\dot{x} = p, \quad \dot{p} = -\sin x + \varepsilon \tilde{k} \sin \phi, \tag{3}$$

where we denote the perturbation phase $\tilde{k}x + \nu t$ as ϕ. Hereafter we shall consider the case when the parameters k and ν have sufficiently large positive values, so that the inequality $d\phi/dt \gg 1$ is satisfied along a particle trajectory, except for some small resonant region, where

$$\frac{d\phi}{dt} = \tilde{k}p - ka\Omega x \sin \Omega t + \nu \simeq 0. \tag{4}$$

Outside this region the particle dynamics is close to integrable and can be described using the averaging technique [13]. According to Eq. (4), the resonant area in phase space is located along the line given by the equation

$$p_{\mathrm{res}} = -\frac{\nu}{k(1 + a \cos \Psi)} + \frac{a\Omega \sin \Psi}{1 + a \cos \Psi} x, \tag{5}$$

where $\Psi = \Omega t$. In order to describe the motion inside the resonant region we derive, using Eqs. (3) and (4), the "pendulumlike" equation for ϕ:

$$\ddot{\phi} - \varepsilon \tilde{k}^2 \sin \phi + f(x,p,\Psi) = 0, \tag{6}$$

where $f(x,p,\Psi)$ is treated as a slowly varying parameter given by the equation

$$f(x,p,\Psi) = ak\Omega(2p \sin \Psi + \Omega x \cos \Psi) + \tilde{k} \sin x. \tag{7}$$

Equation (6) corresponds to the Hamiltonian of the following form:

$$\tilde{H}(\dot{\phi},\phi) = \frac{1}{2}\dot{\phi}^2 + \phi f(x,p,\Psi) + \varepsilon \tilde{k}^2 \cos \phi. \tag{8}$$

If the inequality

$$|f(x,p,\Psi)| < \varepsilon \tilde{k}^2 \tag{9}$$

is satisfied, then the phase portrait corresponding to the Hamiltonian (8) contains an oscillating resonant region in ϕ-$\dot{\phi}$ space, bounded by a separatrix. Falling into resonance (4) corresponds to crossing the separatrix and entering this region. The probability of entering depends on the area of the resonant region and increases with increasing the difference between the left- and right-hand sides of (9). A particle spends any time inside the resonant zone and then leaves it

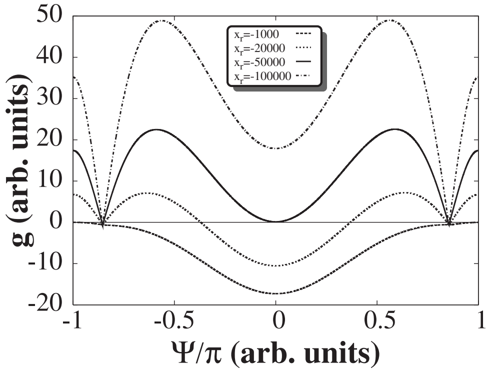

FIG. 1. Function $g(x,\Psi)$ with $x=-1000$, $x=-20\,000$, $x=-50\,000$, and $x=-100\,000$.

with strongly increased or decreased energy. Following [14–16], we derive an approximate formula for the energy jump:

$$\Delta E = -\varepsilon \tilde{k}^* p^* \int_{-\infty}^{\phi^*} \frac{\sin \phi d\phi}{\sqrt{2(\tilde{H} - \phi f^* - \varepsilon \tilde{k}^{*2} \cos \phi)}}, \qquad (10)$$

where $\tilde{k}^*$, f^*, p^*, and ϕ^* are values of $\tilde{k}$, f, p, and ϕ, respectively, when the trajectory hits the resonant region. The value of the integral (10) is extremely sensitive to small changes of initial conditions; therefore, multiple recurrences to the resonant area cause chaotic mixing in phase space [17–19].

In the case of $a=0$ the resonant condition (4) has the simplest form

$$kp + \nu \simeq 0. \qquad (11)$$

It should be emphasized that inequality (9) is fulfilled only if a particle is not far from an extremum of the unperturbed potential. Hence the condition (11) should be replaced by the following one [18]:

$$p(E_{\text{res}}, x = \pi l) = p_{\text{res}} \simeq -\frac{\nu}{k}, \qquad (12)$$

where l is integer. Using Eq. (12) we find the resonant values of the energy:

$$E_{\text{res}} = \frac{\nu^2}{2k^2} \mp 1. \qquad (13)$$

Equation (13) determines locations of the chaotic layers in energy space [11,17,18]. If the chaotic layer induced by resonance (12) coalesces with the near-separatrix chaotic layer, then the chaotic sea formed has a much larger width in the lower half-plane of phase space than in the upper one [11]. It follows from the asymmetry of the condition (12) in momentum space and implies the prevalence of chaos-induced particle flights towards $x=-\infty$.

It is natural to suggest that adiabatic variation of the resonant momentum (12) leads to a gradual displacing of areas of instability in phase space. If the time scale of diffusive mixing inside the chaotic areas is much smaller than the time scale of changing the resonant value of momentum (12), then these areas play the role of dynamical traps for particles,

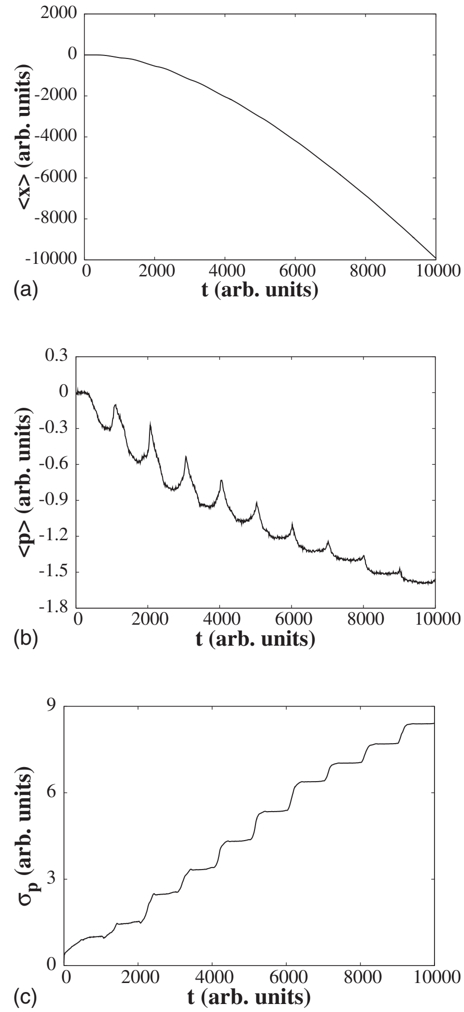

FIG. 2. (a) Mean coordinate, (b) mean momentum, and (c) momentum variance as functions of time.

so-called stochastic layer traps (SLTs) [20,21]. Consequently displacement of a chaotic layer in energy space can be followed by increasing or decreasing of the mean energy of particles belonging to it. As will be shown in this paper, a rather complicated situation occurs if the wave number of the perturbation varies according to the law (2).

Substituting Eq. (5) into Eq. (7), we obtain the expression for the criterion (9) on the resonant line:

$$\left| \frac{a\Omega^2(-a\cos^2\Psi + \cos\Psi + 2a)}{(1 + a\cos\Psi)^2} x + \sin x - \frac{2a\Omega\nu \sin\Psi}{k(1 + a\cos\Psi)^2} \right| \leqslant \varepsilon \tilde{k}. \qquad (14)$$

This inequality holds if $\sin x \simeq 0$ and, subsequently, $x \simeq \pi l$, where l is an integer. When we skip the term $\sim \sin x$, we can rewrite the criterion (14) as follows:

$$g(x,\Psi) = \left| \frac{a\Omega^2(-a\cos^2\Psi + \cos\Psi + 2a)}{(1 + a\cos\Psi)^2} x - \frac{2a\Omega\nu \sin\Psi}{k(1 + a\cos\Psi)^2} \right| - \varepsilon \tilde{k} \leqslant 0. \qquad (15)$$

Figure 1 represents the function $g(x,\Psi)$ with different fixed

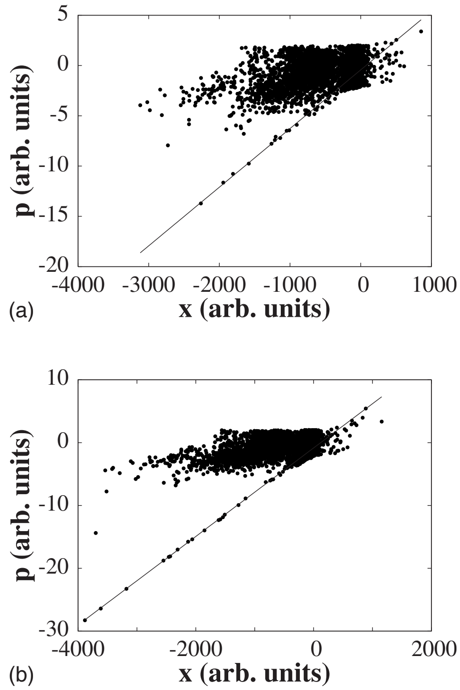

FIG. 3. Particle distribution in phase space at (a) $t=1300$ and (b) $t=1400$. The resonant channel is marked by the line.

values of x. The parameters of the perturbation we used are the following: $\varepsilon=0.04$, $k=12$, $\nu=4$, $a=0.75$, and $\Omega=2\pi/1000$. According to this figure, the criterion (14) is satisfied with $|x|<50\,000$ for the large intervals of Ψ, centered at $2\pi m$, where $m=0,1,2,\ldots$. This implies the existence of those trajectories which, being passed to the resonance area at once, will visit the resonant area repeatedly on the subsequent cycles of pendulum, until the slowly varying phase Ψ remains close to $2\pi m$. Such particles move along the lines described by Eq. (5), towards the point $x=0$ when $\sin\Psi<0$ and from it when $\sin\Psi>0$. The latter ones are capable to perform ballistic flights with increasing velocity. Accelerating flights extend largely the chaotic sea in momentum space, in a similar way as was reported in [22]. It should be noted that we did not find any stable regions by constructing a Poincaré map with $\nu=2\pi$ and $\Omega=2\pi/1000$, even at large values of momentum (of order 10^2).

The occurrence of such ballistic flights is confirmed by numerical simulation. We computed the evolution of the ensemble of particles, initially distributed with Gaussian probability density,

$$\rho(x,p,t=0)=\frac{1}{2\pi\sigma_{0x}\sigma_{0p}}\exp\left(-\frac{x^2}{\sigma_{0x}^2}-\frac{p^2}{\sigma_{0p}^2}\right), \qquad (16)$$

where $\sigma_{0x}=\sigma_{0p}=0.1$. Figure 2 represents the temporal dependence of the mean coordinate, mean momentum, and variance of momentum. The parameters of the perturbation are the same as in Fig. 1. It is shown that there occurs a particle flux directed towards $t\to-\infty$. The mean momentum grows nonmonotonically, and abrupt accelerations are alternating with abrupt slowing down. Acceleration takes place when the slow phase Ψ is close to $2\pi m$, which is consistent with our analysis. Each act of acceleration is followed by a steplike increase of the momentum variance. It should be emphasized that the momentum variance is much larger than the mean momentum, which indicates the presence of particles with very high velocities. Figure 3 shows that accelerating particles form jets along the resonant line (5), which is cut according to the criterion (15). The first significant jet becomes apparent at $t\simeq1300$. Accelerating particles follow the resonant line until $t\simeq1400$ and then leave the resonant zone. It should be noted that strongly accelerated particles never return into the resonant zone again. The formation of later jets is demonstrated in Fig. 4, where instantaneous particle distributions at $t=3200$, $t=5200$, and $t=9200$ are presented. The evolution of the particle cloud is also presented in the media file, which is available at [23].

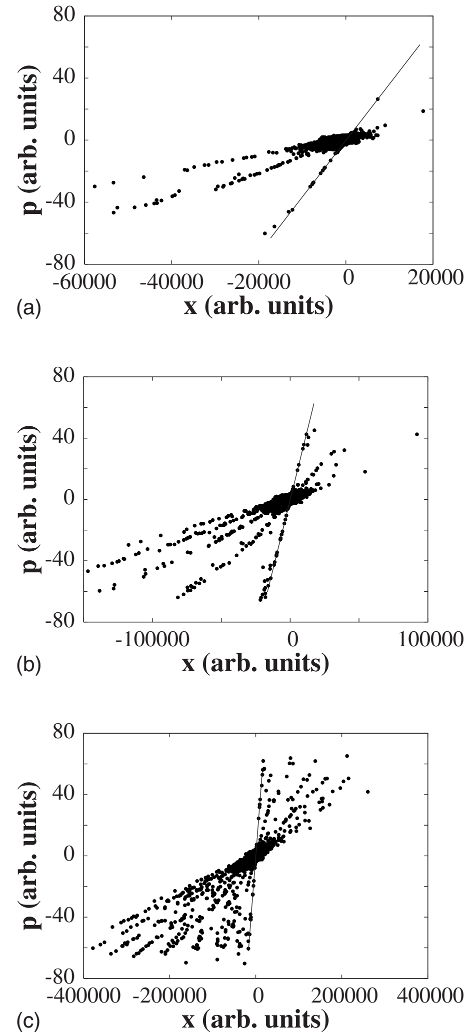

FIG. 4. The same as in Fig. 2 at (a) $t=3200$, (b) $t=5200$, and (c) $t=9200$.

We can distinguish three stages of the evolution of the particle ensemble. At the first stage directed current is activated. Until a particle is not far from $x=0$, the location of the resonant zone is determined by formula (11)—i.e., by the instantaneous value of the ratio $\nu/\tilde{k}(t)$. It infers that the initial particle cloud is placed inside the chaotic layer caused by resonance with $p_{\mathrm{res}}=-\nu/\tilde{k}(t=0)=-4/21$. An adiabatic decrease of k displaces this layer to the separatrix. The time of diffusive mixing inside the chaotic layer is much smaller than $2\pi/\Omega$, so that the particle cloud follows the chaotic layer. At $\Psi=\pi$ the chaotic layer caused by the resonance (4) merges into the near-separatrix chaotic layer, which leads to the occurrence of ballistic particle current towards $x=-\infty$.

The second stage starts when the particle cloud becomes wide enough and some particles are capable of falling into the resonant channels described by Eq. (5). This stage is characterized by fast growth of the momentum variance due to events of giant acceleration. Note that some particles turn around and then perform ballistic flights in the opposite direction. Nevertheless, the number of particles accelerating in the direction $x\to-\infty$ is much larger, and therefore the turned particles give a negligible contribution to the resulting particle flux. Since the resonant channels have a finite length, one can call the zone where they exist the *accelerating zone*.

The third stage is not presented in the figures. It starts when the particle cloud becomes very wide and only a negligible fraction of the particle ensemble remains within the accelerating zone. At this stage the momentum variance achieves saturation and stops increasing.

In conclusion, in this work we demonstrated the effect of giant particle acceleration in the simple space-periodic Hamiltonian system subjected to a slowly modulated external force. The effect arises from the specific topology of resonance (4) in phase space, which permits the capture of a particle into the accelerating channel.

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